ELEN 4810 Homework 1

Analytical Questions

2.23 (a) (1) Suppose that $|x[n]| \leq B$ for all n. Then $|y[n]| = |\cos(\pi n)x[n]| \leq |x[n]| \leq B$. Since every bounded input x produces a bounded output y, the system is **stable**. (2) Since $y[n]$ is a function of $x[n]$ only, the system is **causal**. (3) For arbitrary sequences x_1 and x_2,

$$\begin{aligned} T\{\alpha x_1 + \beta x_2\}[n] &= \cos(\pi n)(\alpha x_1[n] + \beta x_2[n]) \\ &= \alpha \cos(\pi n) x_1[n] + \beta \cos(\pi n) x_2[n] \\ &= \alpha T\{x_1\}[n] + \beta T\{x_2\}[n], \end{aligned}$$

and so the system is **linear**. (4) **Not time invariant**. For a counterexample, consider $x[n] = \delta[n]$. Then $y[0] = \cos(0)x[0] = 1$. But if $y' = T\{x'\}$, with $x'[n] = x[n-1] = \delta[n-1]$, $y'[1] = \cos(\pi)x'[1] = -1 \times 1 = -1 \neq y[0]$. Since there exists inputs $x'[n] = x[n-1]$ whose corresponding outputs $y'[n] \neq y[n-1]$, the system is not time invariant.

(b) (1) Suppose that $|x[n]| \leq B$ for all n. Then $|y[n]| = |x[n^2]| \leq B$, and so y is bounded. Since every bounded input produces a bounded output, the system is **stable**. (2) Notice that $y[-1] = x[(-1)^2] = x[1]$. Since $y[-1]$ depends on a future value $x[1]$, the system is **not causal**. (3) For arbitrary sequences x_1 and x_2,

$$\begin{aligned} T\{\alpha x_1 + \beta x_2\}[n] &= (\alpha x_1 + \beta x_2)[n^2] && (1) \\ &= \alpha x_1[n^2] + \beta x_2[n^2] && (2) \\ &= \alpha T\{x_1\}[n] + \beta T\{x_2\}[n] && (3) \end{aligned}$$

and so the system is **linear**. (4) The system is **not time invariant**. For a counterexample, notice that if $x[n] = \delta[n]$, $T\{x\} = \delta$ as well. But for $x'[n] = \delta[n-2]$, $T\{x'\}[n] = 0$ for all n, and so $T\{\mathcal{D}_2 x\} \neq \mathcal{D}_2 T\{x\}$.

(c) (1) **stable**. Notice that for every n, $|\sum_{k=0}^{\infty} \delta[n-k]| \leq 1$, since at most one of the terms in the summation is nonzero. Hence, $|x[n]\sum_{k=0}^{\infty} \delta[n-k]| \leq |x[n]|$, and every bounded input x produces a bounded output $T\{x\}$. (2) **causal**. $T\{x\}[n]$ depends only on $x[n]$. (3) **linear**. To keep the notation more concise, we can notice that $\sum_{k=0}^{\infty} \delta[n-k] = u[n]$, and simply write

$$\begin{aligned} T\{\alpha x_1 + \beta x_2\}[n] &= (\alpha x_1 + \beta x_2)[n]u[n] && (4) \\ &= \alpha x_1[n]u[n] + \beta x_2[n]u[n] && (5) \\ &= \alpha T\{x_1\}[n] + \beta T\{x_2\}[n]. && (6) \end{aligned}$$

(4) **Not time invariant**. Again, consider $x[n] = \delta[n]$. $T\{x\}[n] = \delta[n]u[n] = \delta[n]$. But $T\{\mathcal{D}_{-1}x\}[n] = \delta[n+1]u[n] = 0 \neq \mathcal{D}_{-1}T\{x\}[n]$.

(d) (1) **Not stable**. For a counterexample, consider the bounded input x with $x[n] = 1$ for all n. Then $T\{x\}[n] = \sum_{k=n-1}^{\infty} 1 = +\infty$ is not bounded. (2) **Not causal**. The output at time n depends on values of $x[k]$ for $k > n$. (3) **linear** over all sequences x for which the output is well-defined. Again,

$$\begin{aligned}
T\{\alpha x_1 + \beta x_2\}[n] &= \sum_{k=n-1}^{\infty} (\alpha x_1[n] + \beta x_2[n]) && (7)\\
&= \alpha \sum_{k=n-1}^{\infty} x_1[n] + \beta \sum_{k=n-1}^{\infty} x_2[n] && (8)\\
&= \alpha T\{x_1\}[n] + \beta T\{x_2\}[n]. && (9)
\end{aligned}$$

(4) **Time invariant**:

$$\begin{aligned}
T\{\mathcal{D}_\ell x\}[n] &= \sum_{k=n-1}^{\infty} (\mathcal{D}_\ell x)[k] && (10)\\
&= \sum_{k=n-1}^{\infty} x[k-\ell] && (11)\\
&= \sum_{k'=n-\ell-1}^{\infty} x[k'] && (12)\\
&= T\{x\}[n-\ell] && (13)\\
&= (\mathcal{D}_\ell T\{x\})[n]. && (14)
\end{aligned}$$

2.43 Notice that $\delta[n] = (1/3)^n u[n] - 1/3 \times (1/3)^{n-1} u[n-1]$. So, the impulse response of the system is $h[n] = g[n] - g[n-1]/3$, and the response of the system to an arbitrary input $x[n]$ is $y[n] = \left(g - \frac{1}{3}\mathcal{D}_1 g\right) * x[n]$.

2.49 We address part (b) first.

(b) Using linearity, we calculate the impulse response of the system. Notice that $\delta[n] = \frac{1}{2}x_1[n] - \frac{1}{2}x_2[n] + x_3[n]$. By linearity, if $x[n] = \delta[n]$, the output $y[n]$ is equal to $y_3[n] - \frac{1}{2}y_2[n] + \frac{1}{2}y_1[n]$. So,

$$h[n] = \begin{cases} 2 & n = -2 \\ 1 & n = -1 \\ -2 & n = 0 \\ 3 & n = 1 \\ 2 & n = 2 \\ 3 & n = 3 \\ 0 & \text{else.} \end{cases} \tag{15}$$

(a) Now, notice that $x_1[n] - x_2[n] = -2\delta[n-2]$. By linearity, the system response to $x[n] =$

$x_1[n] - x_2[n]$ is

$$y[n] = y_1[n] - y_2[n] = \begin{cases} 2 & n = 0 \\ 6 & n = 1 \\ 0 & n = 2 \\ 2 & n = 3 \\ 0 & \text{else} \end{cases} \tag{16}$$

If the system was time invariant, its response to $-2\delta[n-2]$ should be $-2h[n-2]$; this is not the case, and so the system is **not time invariant**.